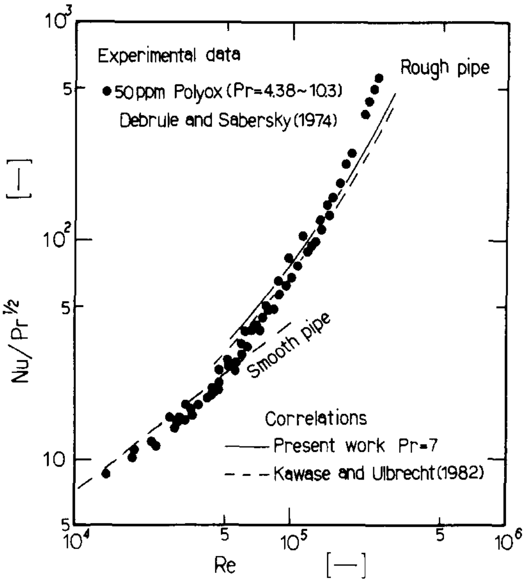

FIG. 2. Comparison of the present model with data for heat and mass transfer (dilute polymer solution under the maximum drag reduction condition).

representation of the heat and mass transfer processes in rough pipes. Unfortunately, more detailed discussion is hampered by the lack of experimental data for dilute polymer solutions and suitable parameters to describe the roughness. A more precise model based on the surface renewal concept must await a greater wealth of knowledge for the bursting phenomena in a rough pipe. Furthermore, it has been known that some of the scatter in the experimental data for dilute polymer solutions is due to degradation of the polymers and entrance effects [21, 22]. Therefore the experimental data carried out taking these effects into consideration is also required.

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THE THERMAL RESISTANCE OF A COMPOSITE HOLLOW SPHERE

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YOYANOVICH *et al.* [1] computed the thermal resistance of a hollow sphere subjected to an arbitrary polar flux. One of their aims was to determine the thermal resistance for isothermal polar regions. One of their assumed flux distributions did give an approximately uniform polar temperature distribution, at least for a small polar angle α and for thick shells. In this note the isothermal problem is solved using an analysis developed by Collins [2].

Figure 1 shows a composite spherical shell composed of shells of external and internal radii (a, b) and (b, c), and thermal

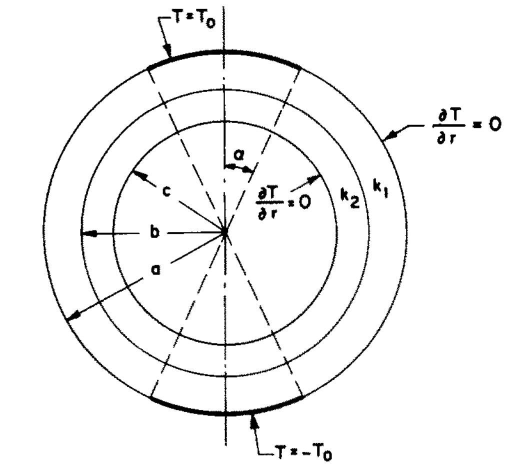

FIG. 1.

conductivities k_1 and k_2, respectively. The inner surface is insulated, the outer surface has caps at temperatures of $\pm T_0$ and its remaining surface is insulated. The static temperature distribution depends only on the spherical coordinates r, θ, and satisfies $\nabla^2 T = 0$. If

$$T \equiv T(r,\theta) = \begin{cases} T_1(r,\theta), & b < r \leqslant a, \\ T_2(r,\theta), & c \leqslant r < b, \end{cases} \tag{1}$$

then the satisfaction of the boundary conditions

$$T_1(a,\theta) = T_0, \quad 0 \leqslant \theta < \alpha, \tag{2}$$

$$T_1(a,\theta) = -T_0, \quad \pi-\alpha < \theta \leqslant \pi, \tag{3}$$

$$\left.\begin{aligned} &\frac{\partial T_1}{\partial r}(a,\theta) = 0, \quad \alpha < \theta < \pi-\alpha; \\ &\frac{\partial T_2}{\partial r}(b,\theta) = 0. \end{aligned}\right. \tag{4}$$

$$\left.\begin{aligned} &T_1(b,\theta) = T_2(b,\theta), \\ &k_1\frac{\partial T_1}{\partial r}(b,\theta) = k_2\frac{\partial T_2}{\partial r}(b,\theta), \end{aligned}\right. \tag{5}$$

leads to the following equations:

$$2\sum_{n=0}^{\infty}(1+H_n)C_nP_n(\cos\theta) = T_0, \qquad 0 \leqslant \theta < \alpha, \tag{6}$$

$$\sum_{n=0}^{\infty}(2n+1)C_nP_n(\cos\theta) = 0, \qquad \alpha < \theta < \pi-\alpha, \tag{7}$$

$$2\sum_{n=0}^{\infty}(1+H_n)C_nP_n(\cos\theta) = -T_0, \quad \pi-\alpha < \theta \leqslant \pi, \tag{8}$$

where $\kappa = k_2/k_1 = 1+\sigma$, $\beta = b/a$, $\gamma = c/b$, $B = \beta^{2n+1}$, $C = \gamma^{2n+1}$ and n takes only 'odd' values

$$2H_n = \frac{(1/n)+[(4n+3)/(n+1)]BC+[\sigma(1-C)/(2n+1)][1-(4n+3)B]}{1-BC+[\sigma(1-C)/(2n+1)][n+(n+1)B]}. \tag{9}$$

These are examples of the 'even' case of equations of the second kind considered by Collins [2]. A simplified version of his reduction of the problem to an integral equation is presented below.

In equations (6)–(8), $C_n = A_n + B_n$, where $n = 0, 1, 2, \ldots$, and A_n and B_n are chosen to satisfy

$$2\sum_{n=0}^{\infty}(1+H_n)(A_n+B_n)P_n(\cos\theta) = T_0, \qquad 0 \leqslant \theta < \alpha, \tag{10}$$

$$\sum_{n=0}^{\infty}(2n+1)A_nP_n(\cos\theta) = 0, \qquad \alpha < \theta \leqslant \pi, \tag{11}$$

$$\sum_{n=0}^{\infty}(2n+1)B_nP_n(\cos\theta) = 0, \qquad 0 \leqslant \theta < \pi-\alpha, \tag{12}$$

$$2\sum_{n=0}^{\infty}(1+H_n)(A_n+B_n)P_n(\cos\theta) = -T_0, \quad \pi-\alpha < \theta \leqslant \pi. \tag{13}$$

Since C_n is zero for even values of n, then $A_{2n} = -B_{2n}$, $A_{2n+1} = B_{2n+1} = \frac{1}{2}C_{2n+1}$. In this way, equations (12) and (13) follow identically from equations (10) and (11) by putting $\theta = \pi - \phi$ and noting that $P_n(-x) = (-)^{n-1}P_n(x)$.

The Legendre polynomials satisfy the equation [3]

$$L(\phi,\theta) \equiv \sum_{k=0}^{\infty}\cos\left(k+\frac{1}{2}\right)\phi P_k(\cos\theta) = \begin{cases} [2(\cos\phi-\cos\theta)]^{-1/2}, & 0 \leqslant \phi < \theta < \pi, \\ 0, & 0 < \theta < \phi < \pi, \end{cases} \tag{14}$$

from which may be deduced

$$\int_0^{\phi}\frac{P_m(\cos\theta)\sin\theta\,d\theta}{[2(\cos\theta-\cos\phi)]^{1/2}} = \frac{2}{2m+1}\sin\left(m+\frac{1}{2}\right)\phi, \tag{15}$$

$$\int_{\phi}^{\pi}\frac{P_m(\cos\theta)\sin\theta\,d\theta}{[2(\cos\phi-\cos\theta)]^{1/2}} = \frac{2}{2m+1}\cos\left(m+\frac{1}{2}\right)\phi. \tag{16}$$

Introducing the two operators $\mathscr{T}_1$ and $\mathscr{T}_2$ defined by

$$\mathscr{T}_1[f(\theta);\theta\to\phi] = \int_0^{\phi}\frac{\sin\theta f(\theta)\,d\theta}{[2(\cos\theta-\cos\phi)]^{1/2}}, \tag{17}$$

$$\mathscr{T}_2[f(\theta);\theta\to\phi] = \int_{\phi}^{\pi}\frac{\sin\theta f(\theta)\,d\theta}{[2(\cos\phi-\cos\theta)]^{1/2}}, \tag{18}$$

and operating on equations (10) and (11) by $(d/d\phi)\mathscr{T}_1$ and $\mathscr{T}_2$, respectively, and using equations (15) and (16) gives

$$2\sum_{n=0}^{\infty}(1+H_n)(A_n+B_n)\cos\left(n+\frac{1}{2}\right)\phi = T_0\cos\frac{\phi}{2}, \quad 0 \leqslant \phi < \alpha, \tag{19}$$

$$\sum_{n=0}^{\infty}A_n\cos\left(n+\frac{1}{2}\right)\phi = 0, \quad \alpha < \phi < \pi. \tag{20}$$

The following function is now introduced

$$j(\phi) = \sum_{n=0}^{\infty}A_n\cos\left(n+\frac{1}{2}\right)\phi, \tag{21}$$

so that equation (20) states that $j(\phi) = 0, \alpha < \phi \leqslant \pi$. One may consider $j(\phi)$ to be an 'even' function defined in $[-\pi,\pi]$; it is non-zero only in $(-\alpha,\alpha)$. Equation (21) gives

$$A_n = \frac{2}{\pi}\int_0^{\alpha}j(\phi)\cos\left(n+\frac{1}{2}\right)\phi\,d\phi. \tag{22}$$

If this is substituted into equation (19), and using $B_n = (-)^{n-1}A_n$ one obtains

$$j(\phi) - \frac{1}{\pi}\int_{-\alpha}^{\alpha}\left\{\frac{1}{2\cos\frac{1}{2}(\theta-\phi)} - M(\theta-\phi)\right\}j(\theta)\,d\theta = \frac{T_0}{2}\cos\frac{\phi}{2}, \quad -\alpha < \phi < \alpha, \tag{23}$$

where

$$M(\theta-\phi) = 2\sum_{n=0}^{\infty} H_{2n+1}\cos\left(2n+\frac{3}{2}\right)(\theta-\phi). \quad (24)$$

The thermal resistance R is defined by $QR = 2T_0$, where Q is the total heat flux, i.e.

$$Q = 2\pi a k_1 \int_0^{\alpha} \sum_{n=0}^{\infty} (4n+3)C_{2n+1}P_{2n+1}(\cos\theta)\sin\theta\,d\theta. \quad (25)$$

This may be expressed in terms of $j(\phi)$ in the form

$$Q = 8ak_1\int_0^{\alpha} j(\phi)\cos\frac{\phi}{2}\,d\phi, \quad (26)$$

and the dimensionless resistance used in ref. [1] is

$$R^* = k_1(a\sin\alpha)R = \frac{T_0\sin\alpha}{4\int_0^{\alpha} j(\phi)\cos\frac{\phi}{2}\,d\phi}. \quad (27)$$

One notes that, in the limit $\alpha \to 0, j(\phi) = T_0/2$ so that $R^* \to \frac{1}{2}$, in agreement with ref. [1].

The only difficulty encountered in the numerical solution of equation (24) occurred in the evaluation of the kernel $M(\theta-\phi)$. Equations (9) and (25) may be rewritten $2H_n = (1/n) + 2H_n^*$ and

$$M(\theta) = -\frac{1}{2}\cos(\theta/2)\ln\left|\tan(\theta/2)\right| - \frac{\pi}{4}\left|\sin(\theta/2)\right| + 2\sum_{n=0}^{\infty} H_{2n+1}^*\cos[(4n+3)\theta/2], \quad (28)$$

where

$$2H_n^* = \frac{[(4n+3)/(n+1)+(1/n)]B\{C-\sigma(1-C)[(n+1)/(2n+1)]\}}{1-BC+[\sigma(1-C)/(2n+1)][n+(n+1)B]}, \quad (29)$$

and B, C are defined after equation (8).

Equation (24) was solved by putting $\theta = \alpha x$, $\phi = \alpha y$, $j(\theta) = g(x)$, and assuming an expansion for $g(x)$ of the form

$$g(x) = \sum_{n=0}^{N} a_n T_n(x), \quad n = \text{even}. \quad (30)$$

The contribution of the logarithmic in $M(\theta-\phi)$ singularity was calculated by using

$$\int_{-1}^{1} \ln\{2|x-y|\}T_m(x)\,dx = \sum_{n=2}^{\infty} b_{m,n}T_n(y), \quad m,n = \text{even}, \quad (31)$$

where

$$b_{m,n} = \frac{1}{2}\left[\frac{1}{n+m+1} - \frac{1}{m+n-1} + \frac{1}{m-n+1} - \frac{1}{m-n-1}\right]. \quad (32)$$

In the single shell case ($\sigma = 0$) it was found that for $0 \leqslant \alpha \leqslant 5°$ and $0 \leqslant \beta \leqslant 0.99$, $g(x)$ could be approximated adequately by taking $N \leqslant 4$ in equation (32). Table 1 shows the dimensionless thermal resistance, to the number of significant figures that were obtained with confidence, i.e. that were reproduced when N in equation (32) was increased. The values obtained by Yoyanovich *et al.* [1] from the heat flux $q = q_0(\cos\theta - \cos\alpha)^{-12}$, are shown for comparison. It is clear that this approximate heat flux generally gives reliable results.

Table 1. Dimensionless thermal resistance. The lower values are the approximate results of Yoyanovich *et al.* [1]

β	α = 0.05	0.10	1.00	5.00
0.0	0.50103 0.5001	0.50187 0.5016	0.51228 0.5127	0.53825 0.5405
0.2	0.50104 0.5001	0.50189 0.5016	0.51248 0.5129	0.53925 0.5415
0.4	0.50112 0.5002	0.50205 0.5017	0.51403 0.5144	0.54693 0.5492
0.6	0.50141 0.5005	0.50263 0.5023	0.51991 0.5203	0.57606 0.5784
0.8	0.50282 0.5019	0.50544 0.5051	0.54799 0.5484	0.71291 0.7157
0.9	0.50671 0.5058	0.51323 0.5129	0.62561 0.6260	1.0754 1.0813
0.92	0.50897 0.5081	0.51775 0.5174	0.67054 0.6710	1.2761 1.2842
0.94	0.51305 0.5121	0.52591 0.5256	0.75130 0.7519	1.624 1.6364
0.96	0.52205 0.5211	0.54391 0.5436	0.92806 0.9290	2.35 2.3662
0.98	0.55306 0.5522	0.60597 0.6056	1.522 1.5248	4.59 4.6171
0.99	0.62481 0.6241	0.7494 0.7491	2.827 2.8204	9.2 9.1720

CONCLUSIONS

It has been shown that the problem of a spherical shell with isothermal polar caps may be formulated as a singular integral equation which may be solved by using a simple series expansion. The results show that the flux distribution used by Yoyanovich *et al.* [1] is adequate for a wide range of problem parameters.

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